INEQUALITIES FOR f-VECTORS OF 4-POLYTOPES

BY

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ABSTRACT

Let V, E, S and F be the number of vertices, edges, subfacets and facets, respectively, of a 4-dimensional convex polytope. In this paper we derive new upper and lower bounds for S in terms of F and V.

1. Introduction

Let P be a d-dimensional convex polytope. The vector $(f_0,f_1,\cdots,f_{d-1})$ is said to be the *f-vector* of P provided P has exactly f_i i-dimensional faces. The f-vectors of 3-dimensional polytopes are those which satisfy the following (see [1, ch. 10]).

(1) $$f_0 - f_1 + f_2 = 2 \quad \text{(Euler's equation)}$$

(2) $$4 \leqq f_0 \leqq 2f_2 - 4$$

(3) $$4 \leqq f_2 \leqq 2f_0 - 4.$$

The f-vectors of 4-dimensional polytopes have not been characterized, however, the following inequalities have been known for some time.

(4) $$2f_0 \leqq f_1 \leqq \binom{f_0}{2}$$

(5) $$2f_3 \leqq f_2 \leqq \binom{f_3}{2}$$

(6) $$f_0 - f_1 + f_2 - f_3 = 0 \quad \text{(Euler's equation).}$$

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In this paper we shall give four new inequalities for the f-vectors of 4-dimensional polytopes.

2. Preliminaries

From here on we shall use the term d-polytope as an abbreviation for "d-dimensional convex polytope", and the term k-face for "k-dimensional face". The terms *vertex*, *edge*, *subfacet* and *facet* will be used for 0-face, 1-face, $(d - 2)$-face and $(d - 1)$-face, respectively, for any d-polytope. For 4-poltyopes we shall use the simpler notation of V, E, S and F for f_0, f_1, f_2 and f_3 respectively.

In this paper we shall deal with triangulations of 2- and 3-spheres, as well as polytopes, as a result there are two theorems about such triangulations that we shall need. The first theorem is that (1), (2) and (3) are true for all triangulations of the 2-sphere. (This the reader may easily verify). The second is the lower bound theorem for triangulated 3-*spheres* [2]:

In any triangulation of the 3-*sphere,*

$$F \geqq 3V - 10$$

where F is the number of 3-*dimensional simplices in the triangulation and V is the number of vertices.*

If $\mathscr{F}$ is a face of a d-polytope P, we shall define a *diagonal* of $\mathscr{F}$ to be a segment joining two vertices of $\mathscr{F}$ that are not joined by an edge of $\mathscr{F}$. Note that a diagonal of $\mathscr{F}$ is also a diagonal of P.

In view of (6) characterizing the f-vectors of 4-polytopes is equivalent to characterizing the triples (V, S, F). These triples form a subset of the lattice points in E^3. In order to facilitate the drawing of diagrams, we shall consider cross sections of this subset determined by fixing the value of V. If we examine what (4) and (5) tell us about each cross section we find that

$$S \leqq \frac{F^2 - F}{2} \tag{7}$$

$$S \geqq 2F \tag{8}$$

$$S \leqq \frac{V^2 - 3V + 2F}{2} \tag{9}$$

$$S \geqq V + F. \tag{10}$$

The inequalities (7) and (8) are merely restatements of (5). Inequality (9) is found by taking the dual of (7), $E \leqq (V^2 - V)/2$, and substituting $V + S - F$ for E. Inequality (10) follows from (8) in the same way.

In Fig. 1 we indicate the region defined by these inequalities for a typical value of V.

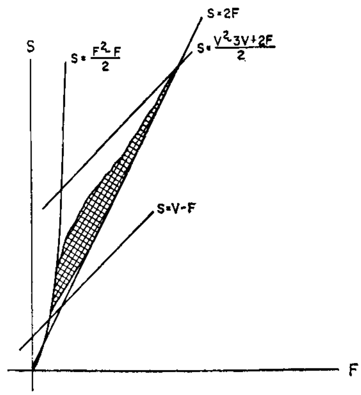

Fig. 1

When one tries to find polytopes with the triples (V, S, F) occuring in this region one seems to be able to fill out only a significantly smaller region (indicated by the crosshatched region in Fig. 1).

Although our new inequalities will not characterize the f-vectors of 4-dimensional polytopes, we shall see that they are better than the previously known unequalities, and that they define a region whose shape is much closer to the shape of the experimentally obtained cross sections of the set of f-vectors.

LEMMA 1. *If P is a 3-polytope with i facets then P has at least $(i^2 - 6i + 8)/8$ diagonals.*

PROOF. For each 2-face of P with n vertices, $n > 3$, we add $n - 3$ diagonals across that 2-face. In this way we have produced a triangulation T of the 2-sphere. If k is the number of diagonals we have added, then T has $i + k$ triangles $(i + k + 4)/2$ vertices and $3(i + k)/2$ edges. This triangulation will then have

$$\binom{\frac{i+k+4}{2}}{2} - \frac{3(k+i)}{2} + k = \frac{i^2 + k^2 + 8 - 6i + 2ik + 2k}{8}$$

diagonals. For any fixed i this expression has minimum value when $k=0$, thus we have at least $(i^2 - 6i + 8)/8$ diagonals.

3. The inequalities

THEOREM 1. *If P is a 4-polytope with V, S and F vertices, subfacets and facets respectively, then*

$$S \leqq \frac{3F}{2} + \frac{V^2 - 3V}{4}$$

and

$$S \leqq \frac{V}{2} + \frac{F^2 + F}{4}.$$

PROOF. Let p_i be the number of facets of P that have i subfacets. Observing that each 5 subfaceted facet has at least one diagonal, we use the following sum to count diagonals of P:

$$p_5 + \sum_{i \geqq 6} \left(\frac{i^2 - 6i + 8}{8}\right) p_i. \tag{11}$$

This sum adds up lower bounds on the number of diagonals of each facet. It is possible that some diagonals are counted twice by this sum, namely those which are diagonals of subfacets and are thus diagonals of two facets.

If we count diagonals of an n-gon on some facet we will count at most $n - 3$ of them. However, each n-gon, $n > 3$, clearly has more than twice this number of diagonals; thus there will be no error due to double counting and (11) is a lower bound on the number of diagonals of P. That is

$$p_5 + \sum_{i \geqq 6} \left(\frac{i^2 - 6i + 8}{8}\right) p_i \leqq \binom{V}{2} - E. \tag{12}$$

We also know that

$$S = \tfrac{1}{2} \sum_{i \geqq 4} i p_i \tag{13}$$

$$p_i \geqq 0 \text{ for all } i. \tag{14}$$

$$\sum_{i \geqq 3} p_i = F \tag{15}$$

We shall combine (13) and (15) to get

$$2S = 4F + \sum_{i \geqq 5} (i - 4) p_i. \tag{16}$$

Now we examine the following linear program: What is the maximum value of $2S$, as expressed in (16), subject to (12) and (14)? We shall consider F, V and E to be constants while the p_i's are our variables. The feasible region of this program will be a polytope in some Euclidean space with a vertex at the origin and vertices on each of the coordinate axes. To find the maximum we examine the function $4F + \sum_{i\geqq 5}(i-4)p_i$ at each vertex.

Case I. The vertex is the origin.

In this case each $p_i = 0$, $i \geqq 5$ and $2S = 4F$.

Case II. A vertex on a coordinate axis.

In this case all but one $p_i = 0$ for $i \geqq 5$.

Case IIa. $p_5 \neq 0$.

The value of $2S$ is $4F + p_5$. From (12) we have $p_5 \leqq \binom{V}{2} - E$. Thus $2S \leqq 4F + \binom{V}{2} - E$.

Case IIb. $p_5 = 0$, $p_i \neq 0$ for some $i > 5$.

The value of $2S$ is $4F + (i-4)p_i$,

from (12) we have $p_i \leqq \dfrac{8}{i^2 - 6i + 8}\left[\binom{V}{2} - E\right]$,

thus
$$\begin{aligned} 2S &\leqq 4F + (i-4)\frac{8}{i^2-6i+8}\left[\binom{V}{2} - E\right] \\ &\leqq 4F + \frac{8}{(i-2)}\left[\binom{V}{2} - E\right] \leqq 4F + 2\binom{V}{2} - 2E. \end{aligned}$$

We conclude that for all values of the p_i's, $2S \leqq 4F + 2\binom{V}{2} - 2E$.

Substituting for E using Euler's equation we obtain

$$\begin{aligned} 2S &\leqq 4F + V^2 - V - 2(V + S - F) \\ 4S &\leqq 6F + V^2 - 3V \\ S &\leqq \frac{3F}{2} + \frac{V^2 - 3V}{4}. \end{aligned}$$

The second inequality follows by duality. Substituting V for F, F for V, S for E and E for S we have

$$E \leqq \frac{3V}{2} + \frac{F^2 - 3F}{4}$$

Eliminating E using Euler's equation gives

$$S \leqq \frac{V}{2} + \frac{F^2 + F}{4}.$$

THEOREM 2. *For all* 4-*dimensional polytopes P*,

$$S \geqq \frac{3V - 10 + 15F}{8} \quad \textit{and} \quad S \geqq \frac{7V - 10 + 11F}{8}.$$

PROOF. We divide each n-gonal subfacet of P into $n - 3$ triangles by adding diagonals across the n-gon. This triangulates the boundaries of the facets. Inside each of these facets we place a new vertex v and then fill out the facet with simplices of the form con $[\{v\} \cup \mathscr{F}]$, where $\mathscr{F}$ is a 2-dimensional face of the facet. In this way we have created a triangulation T of the 3-sphere. Let F^* be the number of tetrahedra of T.

We shall now find an upper bound on F^*. Let us consider any facet of P with i subfacets. By (2) we have that the facet has at most $2i - 4$ vertices and by (3) we have that the facet will have at most $2(2i - 4) - 4$ triangles when its boundary is triangulated.

Now we have that

$$F^* \leqq \sum_{i \geqq 4} (4i - 12)p_i \tag{17}$$

where p_i is the number of facets of P with i subfacets. But

$$\sum_{i \geqq 4} (4i - 12)p_i = 4\Sigma i p_i - 12\Sigma p_i = 8S - 12F.$$

Using the lower bound theorem for triangulated 3-spheres [2] we also have

$$F^* \geqq 3(F + V) - 10. \tag{18}$$

Combining (17) and (18) we have

$$S \geqq \frac{3V - 10 + 15F}{8}.$$

The second inequality is obtained by dualizing and substituting $V + S - F$ for E.

Easy calculation shows that our upper bounds provide better bounds on S than (7) and (9) for all values that F can have. Our lower bounds are better than (8) and (10) for $(V + 10)/3 \leqq F \leqq 3V - 10$. Figure 2 shows the region now defined by all of our inequalities.

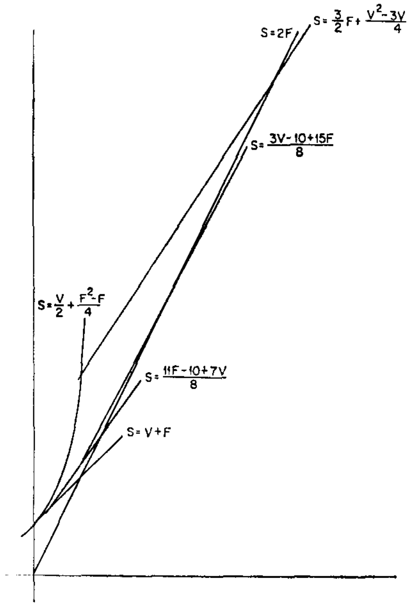

Fig. 2

These inequalities still do not characterize the f-vectors of 4-polytopes. For example the vector $(9, 25, 32, 16)$ satisfies the inequalities yet any polytope with this f-vector would be a simplicial polytope that contradicts the lower bound theorem. One can also find other, less obvious examples to show that our lower bounds are not sharp. The author has not been able to prove that the upper bound are not sharp.

To give an idea of how the cross sectional regions determined by these bounds compare with f-vectors that the author has been able to find, we give a diagram of the region and the f-vectors for $V = 9$ (Fig. 3). The dark border lines are the boundaries determined by our inequalities. The shaded area represents the f-vectors that the author has been able to find.

In general, the upper bounds seem to be sharp only for very large and very small values of F.

One can show that the bound $S \geqq 2F$ is sharp for $F \geqq 3V - 10$, and dually, $S \geqq F$ is sharp for $F \leqq (V - 10)/3$. The lower bounds $S \geqq (11F - 10 + 7V)/8$ and $S \geqq (3V - 10 + 15)/8$ do not appear to be sharp for very many values of F. The author conjectures that $S \geqq (3V - 10 + 7F)/4$ and that this is a sharp bound for most values of F.

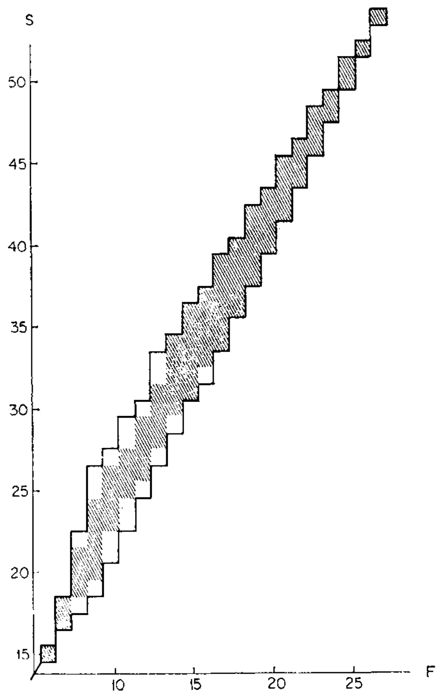

Fig. 3

References

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2. D. Walkup, *The lower bound conjecture for* 3-*and* 4-*manifolds*, Acta Mathematica, **125** (1970), 75–107.

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